BENDING EQUATION.

Assumptions

(1) The material of the beam is homogeneous (material is of same kind through out) and isotropic (elastic properties in all directions are equal.

(2) The value of Young's modulus of Elasticity is the same in tension and compression

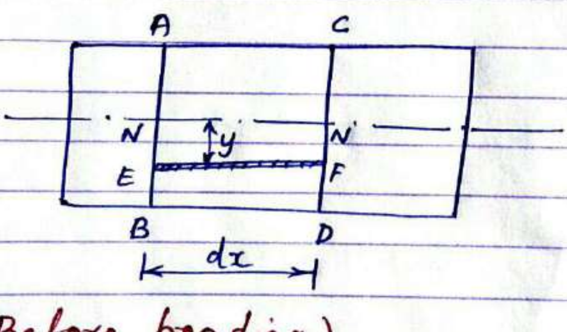
(3) The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.

(4) The radius of curvature is large compared to the cross sectional dimensions of the beam.

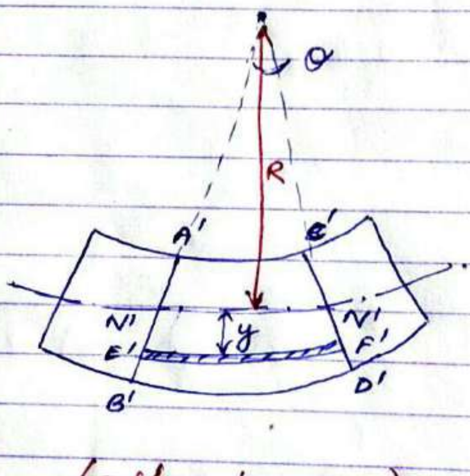
(5) Each layer of the beam is free to expand or contract, independantly of the layer, above or below it.

(6) Transverse sections remains plane before and after bending.

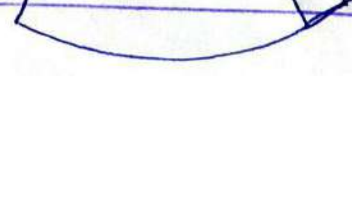
Derivation



(Before bending)



(After bending)



From the fig; $EF = NN = N'N' = dx$

$N'N' = R.\theta$ ($\because$ Arc = Radius x Angle)

||ly $E'F' = (R+y)\theta$

$\therefore$ change in length $= E'F' - EF$

$= (R+y)\theta - R\theta$

$= R\theta + y\theta - R\theta$

$= y\theta$

strain (ε) = $\dfrac{\text{change in length}}{\text{Original length}} = \dfrac{E'F' - EF}{EF}$

$= \dfrac{y\theta}{R\theta}$

$\therefore \varepsilon = \dfrac{y}{R} \longrightarrow (1)$

we have Young's modulus: $E = \dfrac{\sigma}{\varepsilon}$

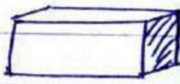
$\therefore \sigma = E.\varepsilon$

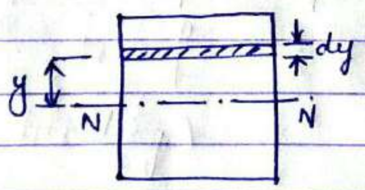
$= E.\dfrac{y}{R}$

Re arranging:

$\dfrac{\sigma}{y} = \dfrac{E}{R} \longrightarrow (2)$

Consider the cross-section of the beam;





Force on the strip having area $dA = \sigma \times dA$ ($\because \sigma = \dfrac{\text{Force}}{\text{Area}}$)

Moment about the neutral axis (dM) = Force x $\perp^{ar}$ distance

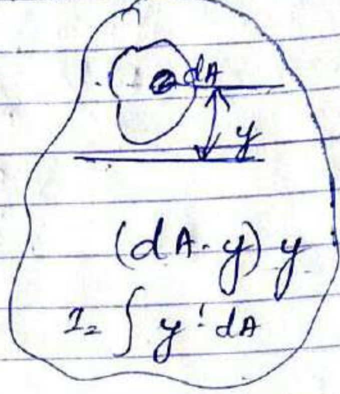
$= (\sigma . dA) \times y$

$$dM = \left(\frac{E}{R} \cdot y \cdot dA\right) \cdot y \qquad \left(\because \text{From } ② \ \sigma = \frac{Ey}{R}\right)$$

Total Bending moment : M is given by integrating the above expression. we get

$$M = \frac{E}{R} \int y^2 \cdot dA.$$

But $\int y^2 \cdot dA = I$ = Moment of Inertia



$$\therefore \quad M = \frac{E}{R} \cdot I$$

$$\frac{M}{I} = \frac{E}{R} \longrightarrow ③$$

Compare ② & ③ : RHS remains the same

$$\therefore \quad \boxed{\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}}$$

where M = Bending moment in N·mm
I = Moment of Inertia in mm^4
E = Young's modulus in N/mm^2
σ = Bending stress in N/mm^2
R = Radius of curvature in mm
y = distance of the layer from Neutral axis in mm

From equation (*i*), it is clear that B.M. at any section is proportional to the distance of the section from the free end.

At $x = 0$ *i.e.*, at B, B.M. = 0

At $x = L$ *i.e.*, at A, B.M. = $W \times L$

Hence B.M. follows the straight line law. The B.M. diagram is shown in Fig. 6.14 (*c*). At point A, take $AC = W \times L$ in the downward direction. Join point B to C.

The shear force and bending moment diagrams for several concentrated loads acting on a cantilever, will be drawn in the similar manner.

Problem 6.1. *A cantilever beam of length 2 m carries the point loads as shown in Fig. 6.15. Draw the shear force and B.M. diagrams for the cantilever beam.*

Sol. Given :

Refer to Fig. 6.15.

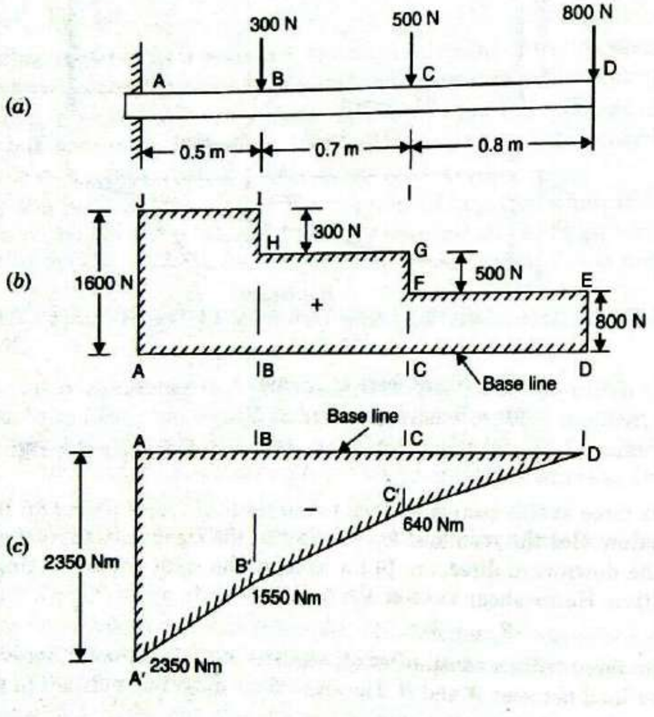


Fig. 6.15

Shear Force Diagram

The shear force at D is + 800 N. This shear force remains constant between D and C. At C, due to point load, the shear force becomes (800 + 500) = 1300 N. Between C and B, the shear force remains 1300 N. At B again, the shear force becomes (1300 + 300) = 1600 N. The shear force between B and A remains constant and equal to 1600 N. Hence the shear force at different points will be as given below :

S.F. at D, $F_D = +800$ N

S.F. at C, $F_C = +800 + 500 = +1300$ N

S.F. at B, $F_B = +800 + 500 + 300 = 1600$ N

S.F. at A, $F_A = +1600$ N.

The shear force, diagram is shown in Fig. 6.15 (*b*) which is drawn as :

Draw a horizontal line AD as base line. On the base line mark the points B and C below the point loads. Take the ordinate $DE = 800$ N in the upward direction. Draw a line EF parallel to AD. The point F is vertically above C. Take vertical line $FG = 500$ N. Through G, draw a horizontal line GH in which point H is vertically above B. Draw vertical line $HI = 300$ N. From I, draw a horizontal line IJ. The point J is vertically above A. This completes the shear force diagram.

Bending Moment Diagram

The bending moment at D is zero :

(*i*) The bending moment at any section between C and D at a distance x and D is given by,

$$M_x = -800 \times x \text{ which follows a straight line law.}$$

At C, the value of $x = 0.8$ m.

∴ B.M. at C, $M_C = -800 \times 0.8 = -640$ Nm.

(*ii*) The B.M. at any section between B and C at a distance x from D is given by (At C, $x = 0.8$ and at B, $x = 0.8 + 0.7 = 1.5$ m. Hence here x varies from 0.8 to 1.5).

$$M_x = -800x - 500(x - 0.8) \qquad \ldots(i)$$

Bending moment between B and C also varies by a straight line law.

B.M. at B is obtained by substituting $x = 1.5$ m in equation (*i*),

$$\therefore \quad M_B = -800 \times 1.5 - 500(1.5 - 0.8)$$
$$= -1200 - 350 = -1550 \text{ Nm.}$$

(*iii*) The B.M. at any section between A and B at a distance x from D is given by (At B, $x = 1.5$ and at A, $x = 2.0$ m. Hence here x varies from 1.5 m to 2.0 m)

$$M_x = -800x - 500(x - 0.8) - 300(x - 1.5) \qquad \ldots(ii)$$

Bending moment between A and B varies by a straight line law.

B.M. at A is obtained by substituting $x = 2.0$ m in equation (*ii*),

$$\therefore \quad M_A = -800 \times 2 - 500(2 - 0.8) - 300(2 - 1.5)$$
$$= -800 \times 2 - 500 \times 1.2 - 300 \times 0.5$$
$$= -1600 - 600 - 150 = -2350 \text{ Nm.}$$

Hence the bending moments at different points will be as given below :

$M_D = 0$

$M_C = -640$ Nm

$M_B = -1550$ Nm

and $M_A = -2350$ Nm.

The bending moment diagram is shown in Fig. 6.15 (*c*) which is drawn as.

Draw a horizontal line AD as a base line and mark the points B and C on this line. Take vertical lines $CC' = 640$ Nm, $BB' = 1550$ Nm and $AA' = 2350$ Nm in the downward direction. Join points D, C', B' and A' by straight lines. This completes the bending moment diagram.

The equation (*i*) shows that the B.M. varies according to the parabolic law. From equation (*i*), we have

At B, $x = 0$ hence $\quad M_B = -(3 \times 0 + 0^2) = 0$

At A, $x = 2$ m hence $\quad M_A = -(3 \times 2 + 2^2) = -10$ kN/m

Now the bending moment diagram is drawn as shown in Fig. 6.18 (*c*). In this diagram, $AA' = 10$ kNm and points A' and B are joined by a parabolic curve.

Problem 6.4. *A cantilever of length 2 m carries a uniformly distributed load of 1.5 kN/m run over the whole length and a point load of 2 kN at a distance of 0.5 m from the free end. Draw the S.F. and B.M. diagrams for the cantilever.*

Sol. Given :

Length, $\quad L = 2$ m

U.D.L., $\quad w = 1.5$ kN/m run

Point load, $\quad W = 2$ kN

Distance of point load from free end = 0.5 m

Refer to Fig. 6.19.

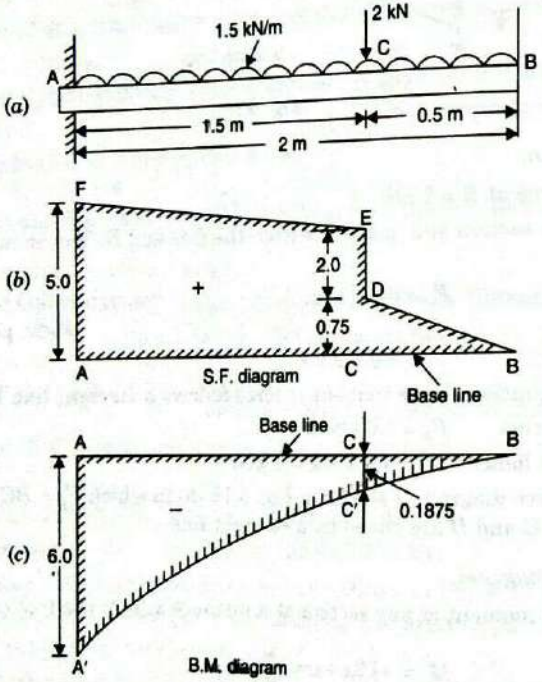


Fig. 6.19

Shear Force Diagram

(*i*) Consider any section between C and B at a distance x from the free end. The shear force at the section is given by,

$$F_x = + w x \qquad \text{(+ve sign is due to downward force on right portion)}$$

$$= 1.5 \times x \qquad \ldots(i)$$

In equation (*i*), x varies from 0 to 0.5. The equation (*i*) shows that shear force varies by a straight line law between B and C.

At B, $x = 0$ hence $\quad F_B = 1.5 \times 0 = 0$

At C, $x = 0.5$ hence $\quad F_C = 1.5 \times 0.5 = 0.75$ kN

(*ii*) Now consider any section between A and C at a distance x from free end B. The shear force at the section is given by

$$F_x = + w.x + 2 \text{ kN} \qquad \text{(+ve sign is due to downward force on right portion of the section)}$$

$$= 1.5x + 2 \qquad \ldots(ii)$$

In equation (*ii*), x varies from 0.5 to 2.0. The equation (*ii*) also shows that shear force varies by a straight line law between A and C.

At C, $x = 0.5$ hence $\quad F_C = 1.5 \times 0.5 + 2 = 2.75$ kN

At A, $x = 2.0$ hence $\quad F_A = 1.5 \times 2.0 + 2 = 5.0$ kN

Now draw the shear force diagram as shown in Fig. 6.19 (*b*) in which $CD = 0.75$ kN, $DE = 2.0$ kN or $CE = 2.75$ kN and $AF = 5.0$ kN. The point B is joined to point D by a straight line whereas the point E is also joined to point F by a straight line.

Bending Moment Diagram

(*i*) The bending moment at any section between C and B at a distance x from the free end B is given by

$$M_x = -(w.x) \cdot \frac{x}{2}$$

$$= -(1.5 \times x) \cdot \frac{x}{2} \qquad (\because \; w = 1.5 \text{ kN/m})$$

$$= -0.75x^2 \qquad \ldots(iii)$$

(The bending moment will be negative as for the right portion of the section the moment at the section is clockwise).

In equation (*iii*), x varies from 0 to 0.5. Equation (*iii*) shows that B.M. varies between C and B by a parabolic law.

At B, $x = 0$ hence $\quad M_B = -0.75 \times 0 = 0$

At C, $x = 0.5$ hence $\quad M_C = -0.75 \times 0.5^2 = -0.1875$ kNm.

(*ii*) The bending moment at any section between A and C at a distance x from the free end B is given by

$$M_x = -(w.x.) \cdot \frac{x}{2} - 2(x - 0.5) = -(1.5 \times x) \cdot \frac{x}{2} - 2(x - 0.5) \qquad (\because \; w = 1.5 \text{ kN/m})$$

$$= -0.75\,x^2 - 2(x - 0.5) \qquad \ldots(iv)$$

In equation (*iv*), x varies from 0.5 to 2.0. Equation (*iv*) shows that B.M. varies by a parabolic law between A and C.

At C, $x = 0.5$ hence $\quad M_C = -0.75 \times 0.5^2 - 2(0.5 - 0.5) = -0.1875$ kN/m

At A, $x = 2.0$ hence $\quad M_A = -0.75 \times 2^2 - 2(2.0 - 0.5)$ kNm $= -3.0 - 3.0 = -6.0$ kNm

Now the bending moment diagram is drawn as shown in Fig. 6.19 (*c*). In this diagram line $CC' = 0.1875$ and $AA' = 6.0$. The points A', C' and B are on parabolic curves.

The values of B.M. at different points are :

At A, $x = 0$ hence $M_A = \frac{w.L}{2}.0 - \frac{w.0}{2} = 0$

At B, $x = L$ hence $M_B = \frac{w.L}{2}.L - \frac{w}{2}.L^2 = 0$

At C, $x = \frac{L}{2}$ hence $M_C = \frac{w.L}{2}.\frac{L}{2} - \frac{w}{2}.\left(\frac{L}{2}\right)^2 = \frac{w.L^2}{4} - \frac{w.L^2}{8} = +\frac{w.L^2}{8}$.

Thus the B.M. increases according to parabolic law from zero at A to $+\frac{w.L^2}{8}$ at the middle point of the beam and from this value the B.M. decreases to zero at B according to the parabolic law.

Now the B.M. diagram is drawn as shown in Fig. 6.27 (c).

Problem 6.9. *Draw the shear force and bending moment diagram for a simply supported beam of length 9 m and carrying a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Also calculate the maximum B.M. on the section.*

Sol. First calculate reactions R_A and R_B.

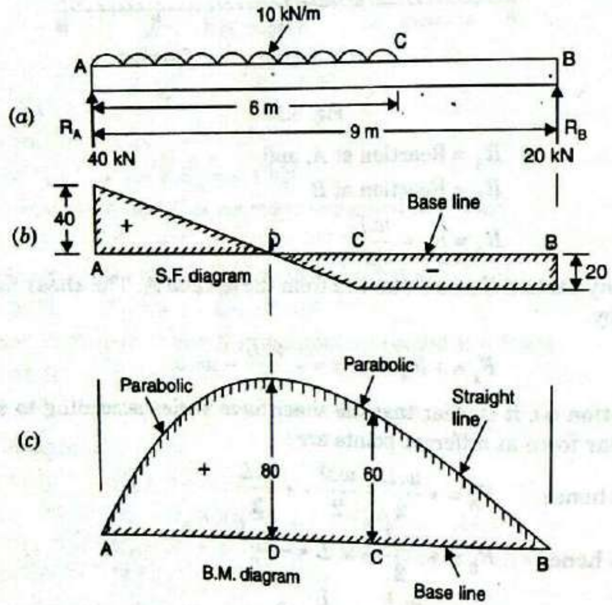


Fig. 6.28

Taking moments of the forces about A, we get

$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$$\therefore \quad R_B = \frac{180}{9} = 20 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B = 10 \times 6 - 20 = 40 \text{ kN}.$$

Shear Force Diagram

Consider any section at a distance x from A between A and C. The shear force at the section is given by,

$$F_x = +R_A - 10x = +40 - 10x \quad \text{...(i)}$$

Equation (*i*) shows that shear force varies by a straight line law between A and C.

At A, $x = 0$ hence $F_A = +40 - 0 = 40$ kN

At C, $x = 6$ m hence $F_C = +40 - 10 \times 6 = -20$ kN

The shear force at A is + 40 kN and at C is – 20 kN. Also shear force between A and C varies by a straight line. This means that somewhere between A and C, the shear force is zero. Let the S.F. is zero at x metre from A. Then substituting the value of S.F. (*i.e.*, F_x) equal to zero in equation (*i*), we get

$$0 = 40 - 10x$$

$$\therefore \quad x = \frac{40}{10} = 4 \text{ m}$$

Hence shear force is zero at a distance 4 m from A.

The shear force is constant between C and B. This equal to – 20 kN.

Now the shear force diagram is drawn as shown in Fig. 6.28 (*b*). In the shear force diagram, distance AD = 4 m. The point D is at a distance 4 m from A.

B.M. Diagram

The B.M. at any section between A and C at a distance x from A is given by,

$$M_x = R_A \times x - 10 . x . \frac{x}{2} = 40x - 5x^2 \quad \text{...(ii)}$$

Equation (*ii*) shows that B.M. varies according to parabolic law between A and C.

At A, $x = 0$ hence $M_A = 40 \times 0 - 5 \times 0 = 0$

At C, $x = 6$ m hence $M_C = 40 \times 6 - 5 \times 6^2 = 240 - 180 = +60$ kNm

At D, $x = 4$ m hence $M_D = 40 \times 4 - 5 \times 4^2 = 160 - 80 = +80$ kNm

The bending moment between C and B varies according to linear law.

B.M. at B is zero whereas at C is 60 kNm.

The bending moment diagram is drawn as shown in Fig. 6.28 (*c*).

Maximum Bending Moment

The B.M. is maximum at a point where shear force changes sign. This means that the point where shear force becomes zero from positive value to the negative or *vice-versa*, the B.M. at that point will be maximum. From the shear force diagram, we know that at point D, the shear force is zero after changing its sign. Hence B.M. is maximum at point D. But the B.M. at D is + 80 kNm.

$\therefore$ **Max. B.M. = + 80 kN. Ans.**

Problem 6.10. *Draw the shear force and B.M. diagrams for a simply supported beam of length 8 m and carrying a uniformly distributed load of 10 kN/m for a distance of 4 m as shown in Fig. 6.29.*

Sol. First calculate the reactions R_A and R_B.

Taking moments of the forces about A, we get

$$R_B \times 8 = 10 \times 4 \times \left(1 + \frac{4}{2}\right) = 120$$

Point of Contraflexure

This point will be between A and B where B.M. is zero after changing its sign. But B.M. at any section at a distance x from A between A and B is given by equation (*iii*) as

$$M_x = 3x - x^2$$

Equation M_x to zero for point of contraflexure, we get

$$0 = 3x - x^2 = x(3 - x)$$

or $\quad 3 - x = 0 \quad$ ($\because$ x cannot be zero as B.M. is not changing sign at this point)

$\therefore \quad x = 3$

Hence point of contraflexure will be at a distance of 3 m from A.

Problem 6.15. *Draw the S.F. and B.M. diagrams for the overhanging beam carrying uniformly distributed load of 2 kN/m over the entire length and a point load of 2 kN as shown in Fig. 6.36. Locate the point of contraflexure.*

Sol. First calculate the reactions R_A and R_B.

Taking moments of all forces about A, we get

$$R_B \times 4 = 2 \times 6 \times 3 + 2 \times 6 = 36 + 12 = 48$$

$$\therefore \quad R_B = \frac{48}{4} = 12 \text{ kN}$$

and $\quad R_A = \text{Total load} - R_B = (2 \times 6 + 2) - 12 = 2 \text{ kN}$

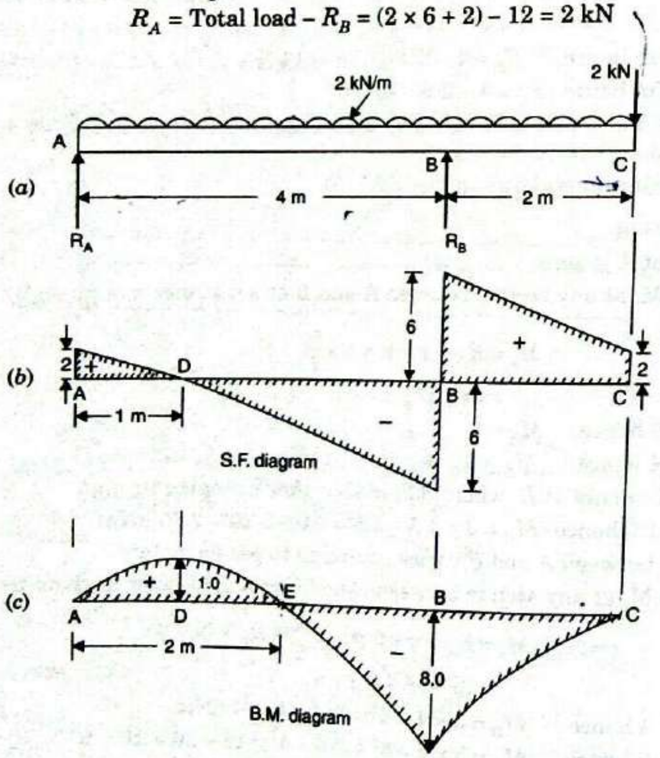


Fig. 6.36

S.F. Diagram

S.F. at $A = + R_A = + 2$ kN

(*i*) The S.F. at any section between A and B at a distance x from A is given by,

$$F_x = + R_A - 2 \times x$$
$$= 2 - 2x \quad \text{...(}i\text{)}$$

At A, $x = 0$ hence $\quad F_A = 2 - 2 \times 0 = 2$ kN

At B, $x = 4$ hence $\quad F_A = 2 - 2 \times 4 = - 6$ kN

The S.F. between A and B varies according to straight line law. At A, S.F. is positive and at B, S.F. is negative. Hence between A and B, S.F. is zero. The point of zero S.F. is obtained by substituting $F_x = 0$ in equation (*i*).

$$\therefore \quad 0 = 2 - 2x \quad \text{or} \quad x = \frac{2}{2} = 1 \text{ m}$$

The S.F. is zero at point D. Hence distance of D from A is 1 m.

(*ii*) The S.F. at any section between B and C at a distance x from A is given by,

$$F_x = + R_A - 2 \times 4 + R_B - 2(x - 4)$$
$$= 2 - 8 + 12 - 2(x - 4) = 6 - 2(x - 4) \quad \text{...(}ii\text{)}$$

At B, $x = 4$ hence $\quad F_B = 6 - 2(4 - 4) = + 6$ kN

At C, $x = 6$ hence $\quad F_C = 6 - 2(6 - 4) = 6 - 4 = 2$ kN

The S.F. diagram is drawn as shown in Fig. 6.36 (*b*).

B.M. Diagram

B.M. at A is zero

(*i*) B.M. at any section between A and B at a distance x from A is given by,

$$M_x = R_A \times x - 2 \times x \times \frac{x}{2} = 2x - x^2 \quad \text{...(}iii\text{)}$$

The above equation shows that the B.M. between A and B varies according to parabolic law.

At A, $x = 0$ hence $\quad M_A = 0$

At B, $x = 4$ hence $\quad M_B = 2 \times 4 - 4^2 = - 8$ kNm

Max. B.M. is at D where S.F. is zero after changing sign

At D, $x = 1$ hence $\quad M_D = 2 \times 1 - 1^2 = 1$ kNm

The B.M. at C is zero. The B.M. also varies between B and C according to parabolic law. Now the B.M. diagram is drawn as shown in Fig. 6.36 (*c*).

Point of Contraflexure

This point is at E between A and B, where B.M. is zero after changing its sign. The distance of E from A is obtained by putting $M_x = 0$ in equation (*iii*).

$$\therefore \quad 0 = 2x - x^2 = x(2 - x)$$

$$2 - x = 0$$

and $\quad$ x = **2 m. Ans.**

Problem 6.16. *A beam of length 12 m is simply supported at two supports which are 8 m apart, with an overhang of 2 m on each side as shown in Fig. 6.37. The beam carries a concentrated load of 1000 N at each end. Draw S.F. and B.M. diagrams.*

Columns and Struts

19.1. INTRODUCTION

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as *column*, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as *strut*. Examples of struts are : connecting rods, piston rods etc.

19.2. FAILURE OF A COLUMN

The failure of a column takes place due to the anyone of the following stresses set up in the columns :

(*i*) Direct compressive stresses,

(*ii*) Buckling stresses, and

(*iii*) Combined of direct compressive and buckling stresses.

19.2.1. Failure of a Short Column. A short column of uniform cross-sectional area A, subjected to an axial compressive load P, is shown in Fig. 19.1. The compressive stress induced is given by

$$p = \frac{P}{A}$$

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

Let P_c = Crushing load,

σ_c = Crushing stress, and

A = Area of cross-section.

Then $$\sigma_c = \frac{P_c}{A}.$$

All short columns fail due to crushing.

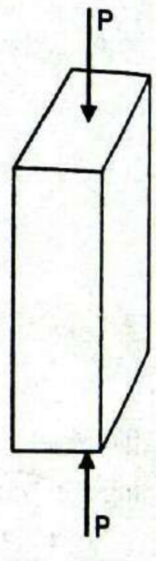


Fig. 19.1

19.2.2. Failure of a Long Column. A long column of uniform cross-sectional area A and of length l, subjected to an axial compressive load P, is shown in Fig. 19.2. A column is known as long column if the length of the column in comparison to its lateral dimensions, is very large. Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown in Fig. 19.2. The load at which the column just buckles, is known as *buckling load or critical just or crippling load*. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.

Refer to Fig. 19.2.

Let l = Length of a long column

P = Load (compressive) at which the column has just buckled

A = Cross-sectional area of the column

e = Maximum bending of the column at the centre

σ_0 = Stress due to direct load $= \dfrac{P}{A}$

σ_b = Stress due to bending at the centre of the column $= \dfrac{P \times e}{Z}$

where Z = Section modulus about the axis of bending.

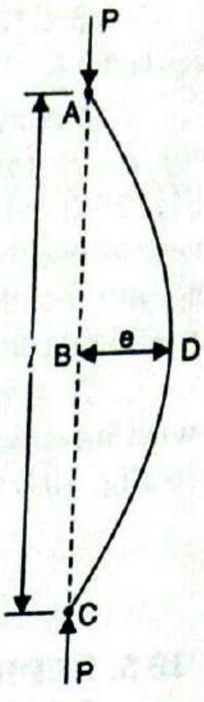


Fig. 19.2

The extreme stresses on the mid-section are given by

Maximum stress $= \sigma_0 + \sigma_b$

and Minimum stress $= \sigma_0 - \sigma_b$.

The column will fail when maximum stress (*i.e.*, $\sigma_0 + \sigma_b$) is more than the crushing stress σ_c. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

19.3. ASSUMPTIONS MADE IN THE EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory :

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

19.4. END CONDITIONS FOR LONG COLUMNS

In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns take place entirely due to buckling (or bending). The following four types of end conditions of the columns are important :

1. Both the ends of the column are hinged (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

19.9. EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column,

l = Actual length of the column, and

P = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^2}. \quad \text{...(19.5)}$$

S.No.	End conditions of column	Crippling load in terms of		Relation between effective length and actual length
		Actual length	Effective length	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{2\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia *i.e.*, I_{xx} and I_{yy}.

The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

RANKINE'S FORMULA
$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \quad \ldots(i)$$

where P = Crippling load by Rankine's formula

P_C = Crushing load = $\sigma_c \times A$

σ_c = Ultimate crushing stress

A = Area of cross-section

P_E = Crippling load by Euler's formula

$= \frac{\pi^2 EI}{L_e^2}$, in which L_e = Effective length

For a given column material the crushing stress σ_c is a constant. Hence the crushing load P_C (which is equal to $\sigma_c \times A$) will also be constant for a given cross-sectional area of the column. In equation (*i*), P_C is constant and hence value of P depends upon the value of P_E. But for a given column material and given cross-sectional area, the value of P_E depends upon the effective length of the column.

(*i*) *If the column is a short*, which means the value of L_e is small, then the value of P_E will be large. Hence the value of $\frac{1}{P_E}$ will be small enough and is negligible as compared to the value of $\frac{1}{P_C}$. Neglecting the value of $\frac{1}{P_E}$ in equation (*i*), we get

$$\frac{1}{P} \to \frac{1}{P_C} \quad \text{or} \quad P \to P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. In Art. 19.2.1 also we have seen that short columns fail due to crushing.

(*ii*) *If the column is long*, which means the value of L_e is large. Then the value of P_E will be small and the value of $\frac{1}{P_E}$ will be large enough compared with $\frac{1}{P_C}$. Hence the value of $\frac{1}{P_C}$ may be neglected in equation (*i*).

$$\therefore \quad \frac{1}{P} = \frac{1}{P_E} \quad \text{or} \quad P \to P_E.$$

Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C \cdot P_E}$.

Taking reciprocal to both sides, we have

$$P = \frac{P_C \cdot P_E}{P_E + P_C} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

(Dividing the numerator and denominator by P_E)

$$= \frac{\sigma_c \times A}{1 + \frac{\sigma_c \cdot A}{\left(\frac{\pi^2 EI}{L_e^2}\right)}} \quad \left(\because P_c = \sigma_c \cdot A \text{ and } P_E = \frac{\pi^2 EI}{L_e^2}\right)$$

But $I = Ak^2$, where k = least radius of gyration

$\therefore$ The above equation becomes as

$$P = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot L_e^2}{\pi^2 E \cdot Ak^2}} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c}{\pi^2 E} \cdot \left(\frac{L_e}{k}\right)^2}$$

$$= \frac{\sigma_c \cdot A}{1 + a \cdot \left(\frac{L_e}{k}\right)^2} \quad \ldots(19.9)$$

where $a = \frac{\sigma_c}{\pi^2 E}$ and is known as Rankine's constant.

The equation (19.9) gives crippling load by Rankine's formula. As the Rankine formula is empirical formula, the value of '*a*' is taken from the results of the experiments and is not calculated from the values of σ_c and E.

The values of σ_c and a for different columns material are given below in Table 19.2.

TABLE 19.2

S. No.	*Material*	σ_c *in N/mm²*	a
1.	Wrought Iron	250	$\frac{1}{9000}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Mild Steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

Problem 19.13. *The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of σ_c = 550 N/mm² and $a = \frac{1}{1600}$ in Rankine's formula.*

Sol. Given :

External dia., D = 5 cm

Internal dia., d = 4 cm

$\therefore$ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$

∴ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, l = 3 m = 3000 mm

As both the ends are fixed,

∴ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, $\sigma_c = 550$ N/mm²

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1+\left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1+\frac{1}{1600}\times\left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \textbf{123750 N. Ans.}$$

Problem 19.14. *A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 kN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take $\sigma_c = 550\ N/mm^2$ and $a = \frac{1}{1600}$ in Rankine's formula.* (AMIE, Winter 1983)

Sol. Given :

Length of column, l = 4 m = 4000 mm

End conditions = Both ends fixed

∴ Effective length, $L_e = \frac{l}{2} = \frac{4000}{2} = 2000$ mm

Safe load, = 250 kN

Factor of safety, = 5

Let External dia., = D

Internal dia. = $0.8 \times D$

Crushing stress, $\sigma_c = 550$ N/mm²

Value of 'a' $= \frac{1}{1600}$ in Rankine's formula

Now factor of safety $= \frac{\text{Cirippling load}}{\text{Safe load}}$ or $5 = \frac{\text{Cirippling load}}{250}$

∴ Crippling load, $P = 5 \times 250 = 1250$ kN = 1250000 N

Area of column, $A = \frac{\pi}{4}[D^2 - (0.8D)^2]$

$= \frac{\pi}{4}[D^2 - 0.64D^2] = \frac{\pi}{4} \times 0.36D^2 = \pi \times 0.09D^2$

Moment of Inertia, $I = \frac{\pi}{64}[D^4 - (0.8D^4] = \frac{\pi}{64}[D^4 - 0.4096D^4]$

$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$

But $I = A \times k^2$, where k is radius of gyration

$$\therefore \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{L_e}{k}\right)^2}$

or
$$1250000 = \frac{550 \times \pi \times 0.09\ D^2}{1 + \frac{1}{1600}\times\left(\frac{2000}{0.32D}\right)^2} \quad (\because\ A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1+\frac{24414}{D^2}} \quad \text{or} \quad 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

or $8038D^2 + 8038 \times 24414 = D^4$ or $D^4 - 8038D^2 - 8038 \times 24414 = 0$

or $D^4 - 8038\ D^2 - 196239700 = 0.$

The above equations is a quadratic equation in D^2. The solution is

$$\therefore \quad D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2} \quad \left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \text{ (The other root is not possible)}$$

$= 18592.5$ mm²

∴ $D = \sqrt{18592.5} = 136.3$ mm

∴ External diameter = **136.3 mm. Ans.**

Internal diameter = 0.8 × 1363 = **109 mm. Ans.**

Problem 19.15. *A 1.5 m long column has a circular cross-section of 5 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using :*

(a) Rankine's formula, take yield stress, $\sigma_c = 560\ N/mm^2$ and $a = \frac{1}{1600}$ for pinned ends.

(b) Euler's formula, Young's modulus for C.I. = $1.2 \times 10^5\ N/mm^2$. (AMIE, Summer 1976)

Sol. Given :

Length, l = 1.5 m = 1500 mm

Diameter, d = 5 cm

Problem 19.18. *A hollow cast iron column 200 mm outside diameter and 150 mm inside diameter, 8 m long has both ends fixed. It is subjected to an axial compressive load. Taking a factor of safety as 6, $\sigma_c = 560\ N/mm^2$, $a = \frac{1}{1600}$, determine the safe Rankine load.*

(AMIE, Summer 1990)

Sol. Given :

External dia., $D = 200$ mm

Internal dia., $d = 150$ mm

Length, $l = 8$ m $= 8000$ mm

End conditions = Both the ends are fixed

Crushing stress, $\sigma_c = 560\ \text{N/mm}^2$

Rankine's constant, $a = \frac{1}{1600}$

Safety factor = 6

Area of cross-section, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - 150^2)$

$= \frac{\pi}{4}(40000 - 22500) = 13744\ \text{mm}^2$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(200^4 - 150^4)$

$= \frac{\pi}{64}(1600000000 - 506250000) = 53689000\ \text{mm}^4$

Least radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689000}{13744}} = 62.5$ mm

Let P = Crippling load by Rankine formula.

Using equation (19.9), $P = \dfrac{\sigma_c \times A}{1 + a\left(\frac{L_e}{k}\right)^2}$

where L_e = Effective length $= \frac{l}{2} = \frac{8000}{2} = 4000$ mm

$\therefore$ $$P = \frac{560 \times 13744}{1 + \frac{1}{1600} \times \left(\frac{4000}{62.5}\right)^2}$$

$$= \frac{7696640}{1 + 2.56} = \frac{7696640}{3.56} = 2161977\ \text{N} = 2161.977\ \text{kN}$$

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{2161.977}{6} =$ **360.3295 kN. Ans.**

Problem 19.19. *A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both ends. Calculate the safe load by Rankine's formula using a factor of safety of 4. Calculate the slenderness ratio and the ratio of Euler's and Rankine's critical loads. Take $\sigma_c = 550\ N/mm^2$, $a = \frac{1}{1600}$ in Rankine's formula and $E = 9.4 \times 10^4\ N/mm^2$.*

(AMIE, Winter 1979 ; Annamalai University, 1991)

Sol. Given :

Outside diameter, $D = 200$ mm

Thickness, $t = 20$ mm

$\therefore$ Inside diameter, $d = D - 2 \times t = 200 - 2 \times 20 = 160$ mm

Area, $A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - 160^2) = 11310\ \text{mm}^2$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(200^4 - 160^4) = 46370000\ \text{mm}^4$

And the least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{46370000}{11310}} = 64\ \text{mm}$$

Length of column, $l = 4.5$ m $= 4500$ mm

End condition = Both the ends are fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{4500}{2} = 2250$ mm

Factor of safety = 4

Value of $\sigma_c = 550\ \text{N/mm}^2$

Value of $a = \frac{1}{1600}$

Value of $E = 9.4 \times 10^4\ \text{N/mm}^2$.

(i) Slenderness ratio

Using equation (19.8), we get

Slenderness ratio $= \frac{l}{k} = \frac{4500}{64} =$ **70.30. Ans.**

(ii) Safe load by Rankine's formula

Let P = Crippling load by Rankine's formula

Using equation (19.9), $P = \dfrac{\sigma_c \times A}{1 + a\left(\frac{L_e}{k}\right)^2} = \dfrac{550 \times 11310}{1 + \frac{1}{1600}\left(\frac{2250}{64}\right)^2} = 351100$ N

$\therefore$ Safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{3511000}{4} =$ **8777 N. Ans.**

(iii) Ratio of Euler's and Rankine's critical loads

Let P_E = Euler's critical load

Euler's critical load is given by equation (19.5)

$$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 9.4 \times 10^4 \times 46370000}{2250^2} = 849770\ \text{N}$$

$\therefore$ $\dfrac{\text{Euler's critical load}}{\text{Rankine's critical load}} = \dfrac{P_E}{P}$

$= \dfrac{8497700}{3511000}$ ($\because$ $P = 3511000$ N)

= 2.42. Ans.

Using the equation (16.22),

$$\sigma = \frac{3Wl}{2nbt^2} \quad \text{or} \quad 100 = \frac{3 \times 2000 \times 1000}{2 \times n \times 50 \times 10^2}$$

or

$$n = \frac{3 \times 2000 \times 1000}{100 \times 2 \times 50 \times 100} = 6. \quad \textbf{Ans.}$$

Deflection under load

Using equation (16.23),

$$\delta = \frac{\sigma \times l^2}{4E \times t} = \frac{100 \times 1000^2}{4 \times 2.1 \times 10^5 \times 10} = \textbf{11.9 mm.} \quad \textbf{Ans.}$$

16.14.2. Helical Springs. Helical springs are the thick spring wires coiled into a helix. They are of two types :

1. Close-coiled helical springs and
2. Open coiled helical springs.

Close-coiled helical springs. Close-coiled helical springs are the springs in which helix angle is very small or in other words the pitch between two adjacent turns is small. A close-coiled helical spring carrying an axial load is shown in Fig. 16.13. As the helix angle in case of close-coiled helical springs are small, hence the bending effect on the spring is ignored and we assume that the coils of a close-coiled helical springs are to stand purely torsional stresses.

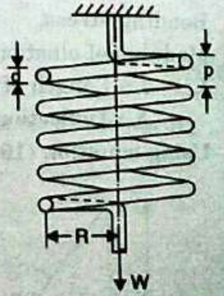


Fig. 16.13

Expression for max. shear stress induced in wire. Fig. 16.13 shows a close-coiled helical spring subjected to an axial load.

Let d = Diameter of spring wire
p = Pitch of the helical spring
n = Number of coils
R = Mean radius of spring coil
W = Axial load on spring
C = Modulus of rigidity
τ = Max. shear stress induced in the wire
θ = Angle of twist in spring wire, and
δ = Deflection of spring due to axial load
l = Length of wire.

Now twisting moment on the wire,

$$T = W \times R \quad ...(i)$$

But twisting moment is also given by

$$T = \frac{\pi}{16} \tau d^3 \quad ...(ii)$$

Equating equations (*i*) and (*ii*), we get

$$W \times R = \frac{\pi}{16} \tau d^3 \quad \text{or} \quad \tau = \frac{16W \times R}{\pi d^3} \quad ...(16.24)$$

Equation (16.24) gives the max. shear stress induced in the wire.

Expression for deflection of spring

Now length of one coil = πD or $2\pi R$

∴ Total length of the wire = Length of one coil × No. of coils or $l = 2\pi R \times n$.

As the every section of the wire is subjected to torsion, hence the strain energy stored by the spring due to torsion is given by equation (16.20).

∴ Strain energy stored by the spring,

$$U = \frac{\tau^2}{4C} \cdot \text{Volume} = \frac{\tau^2}{4C} \cdot \text{Volume}$$

$$= \left(\frac{16W.R}{\pi d^3}\right)^2 \times \frac{1}{4C} \times \left(\frac{\pi}{4} d^2 \times 2\pi R.n\right)$$

$$\left(\because \tau = \frac{16WR}{\pi d^3} \text{ and Volume} = \frac{\pi}{4} d^2 \times \text{Total length of wire}\right)$$

$$= \frac{32W^2R^2}{Cd^4} . R.n = \frac{32W^2R^3 . n}{Cd^4} \quad ...(16.25)$$

Work done on the spring = Average load × Deflection

$$= \tfrac{1}{2} W \times \delta \quad (\because \text{ Deflection} = \delta)$$

Equating the work done on spring to the energy stored, we get

$$\tfrac{1}{2} W.\delta = \frac{32W^2R^3 . n}{Cd^4}$$

$$\therefore \quad \delta = \frac{64WR^3n}{Cd^4} \quad ...(16.26)$$

Expression for stiffness of spring

The stiffness of spring,

s = Load per unit deflection

$$= \frac{W}{\delta} = \frac{W}{\dfrac{64 . WR^3 . n}{Cd^4}} = \frac{Cd^4}{64 . R^3 . n} \quad ...(16.27)$$

Note. The solid length of the spring means the distance between the coils when the coils are touching each other. There is no gap between the coils. The solid length is given by

$$\text{Solid length} = \text{Number of coils} \times \text{Dia. of wire} = n \times d \quad ...(16.28)$$

Problem 16.35. *A closely coiled helical spring is to carry a load of 500 N. Its mean coil diameter is to be 10 times that of the wire diameter. Calculate these diameters if the maximum shear stress in the material of the spring is to be 80 N/mm².*

(AMIE, Summer 1985)

Sol. Given :

Load on spring, $W = 500$ N

Max. shear stress, $\tau = 80$ N/mm²

Let d = Diameter of wire

D = Mean diameter of coil

∴ $D = 10\,d$.

Using equation (16.24), $\tau = \dfrac{16WR}{\pi d^3}$

Stiffness of the spring

$$\text{Stiffness} = \frac{\text{Load on spring}}{\text{Deflection of spring}} = \frac{200}{38.095} = \mathbf{5.25\ N/mm.}\quad \textbf{Ans.}$$

Frequency of free vibration

$$\delta = 38.095 \text{ mm} = 3.8096 \text{ cm}$$

Using the relation, $\tau = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{981}{3.8095}} = \mathbf{2.55\ cycles/sec.}$ **Ans.**

Problem 16.39. *A closely coiled helical spring of mean diameter 20 cm is made of 3 cm diameter rod and has 16 turns. A weight of 3 kN is dropped on this spring. Find the height by which the weight should be dropped before striking the spring so that the spring may be compressed by 18 cm. Take $C = 8 \times 10^4$ N/mm².*

Sol. Given :

Mean dia. of coil, $D = 20 \text{ cm} = 200 \text{ mm}$

∴ Mean radius of coil, $R = \frac{200}{2} = 100 \text{ mm}$

Dia. of spring rod, $d = 3 \text{ cm} = 30 \text{ mm}$

Number of turns, $n = 16$

Weight dropped, $W = 3 \text{ kN} = 3000 \text{ N}$

Compression of the spring, $\delta = 18 \text{ cm} = 180 \text{ mm}$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let h = Height through which the weight W is dropped

W = Gradually applied load which produces the compression of spring equal to 180 mm.

Now using equation (16.26),

$$\delta = \frac{64W.R^3.n}{Cd^4}$$

or
$$180 = \frac{64 \times W \times 100^3 \times 16}{8 \times 10^4 \times 30^4}$$

or
$$W = \frac{180 \times 8 \times 10^4 \times 30^4}{64 \times 100^3 \times 16} = 11390 \text{ N}$$

Work done by the falling weight on spring

$$= \text{Weight falling } (h + \delta) = 3000\,(h + 180) \text{ N-mm}$$

Energy stored in the spring $= \frac{1}{2}W \times \delta$

$$= \tfrac{1}{2} \times 11390 \times 180 = 1025100 \text{ N-mm.}$$

Equating the work done by the falling weight on the spring to the energy stored in the spring, we get

$$3000(h + 180) = 1025100$$

or
$$h + 180 = \frac{1025100}{3000} = 341.7 \text{ mm}$$

∴
$$h = 341.7 - 180 = \mathbf{161.7\ mm.}\quad \textbf{Ans.}$$

Problem 16.40. *The stiffness of a close-coiled helical spring is 1.5 N/mm of compression under a maximum load of 60 N. The maximum shearing stress produced in the wire of the*

spring is 125 N/mm². The solid length of the spring (when the coils are touching) is given as 5 cm. Find : (i) diameter of wire, (ii) mean diameter of the coils and (iii) number of coils required. Take $C = 4.5 \times 10^4$ N/mm².

Sol. Given :

Stiffness of spring, $s = 1.5 \text{ N/mm}$

Load on spring, $W = 60 \text{ N}$

Maximum shear stress, $\tau = 125 \text{ N/mm}^2$

Solid length of spring, $= 5 \text{ cm} = 50 \text{ mm}$

Modulus of rigidity, $C = 4.5 \times 10^4 \text{ N/mm}^2$.

Let d = Diameter of wire,

D = Mean dia. of coil, and

R = Mean radius of coil $= \frac{D}{2}$

n = Number of coils.

Using equation (16.27),

$$s = \frac{Cd^4}{64.R^3.n} \quad \text{or} \quad 1.5 = \frac{4.5 \times 10^4 \times d^4}{64 \times R^3 \times n}$$

∴
$$d^4 = \frac{1.5 \times 64 \times R^3 \times n}{4.5 \times 10^4} = 0.002133R^3 \times n \qquad ...(i)$$

Using equation (16.24),

$$\tau = \frac{16W \times R}{\pi d^3} \quad \text{or} \quad 125 = \frac{16 \times 60 \times R}{\pi d^3}$$

∴
$$R = \frac{125 \times \pi d^3}{16 \times 60} = 0.40906d^3 \qquad ...(ii)$$

Substituting the value of R in equation (i), we get

$$d^4 = 0.002133 \times (0.40906d^3)^3 \times n$$
$$= 0.002133 \times (0.40906^3) \times d^9 \times n = 0.00014599 \times d^9 \times n$$

or
$$\frac{d^9.n}{d^4} = \frac{1}{0.00014599} \quad \text{or} \quad d^5.n = \frac{1}{0.00014599} \qquad ...(iii)$$

Now using equation (16.28),

Solid length $= n \times d$ or $50 = n \times d$

∴
$$n = \frac{50}{d} \qquad ...(iv)$$

Substituting this value of n in equation (iii), we get

$$d^5 \times \frac{50}{d} = \frac{1}{0.00014599}$$

or
$$d^4 = \frac{1}{0.00014599} \times \frac{1}{50} = 136.99$$

∴
$$d = (136.99)^{1/4} = \mathbf{3.42\ mm.}\quad \textbf{Ans.}$$

Substituting this value in equation (iv)

$$n = \frac{50}{d} = \frac{50}{3.42} = \mathbf{14.62\ say\ 15.}\quad \textbf{Ans.}$$

Also from equation (*ii*),

$$R = 0.40906\, d^3 = 0.40906 \times (3.42)^3 = 16.36 \text{ mm}$$

i.e., Mean dia. of coil, $D = 2R = 2 \times 16.36 =$ **32.72 mm. Ans.**

Problem 16.41. *A close-coiled helical spring has a stiffness of 10 N/mm. Its length when fully compressed, with adjacent coils touching each other is 40 cm. The modulus of rigidity of the material of the spring is 0.8×10^5 N/mm².*

(i) Determine the wire diameter and mean coil diameter if their ratio is $\frac{1}{10}$.

(ii) If the gap between any two adjacent coil is 0.2 cm, what maximum load can be applied before the spring becomes solid, i.e., adjacent coils touch ?

(iii) What is the corresponding maximum shear stress in the spring ? (AMIE, May 1976)

Sol. Given :

Stiffness of spring, $s = 10$ N/mm

Length of spring when fully compressed *i.e.*, solid length

$= 40$ cm $= 400$ mm

Modulus of rigidity, $C = 0.8 \times 10^5$ N/mm²

Let d = Diameter of wire of spring

D = Mean coil diameter

n = Number of turns

W = Maximum load applied when spring becomes solid

τ = Maximum shear stress induced in the wire.

Now $\frac{d}{D} = \frac{1}{10}$

Gap between any two adjacent coil = 0.2 cm = 2.0 mm

∴ Total gap in coils = Gap between two adjacent coil × Number of turns

$= 2 \times n$ mm.

When spring is fully compressed, there is no gap in the coils and hence maximum compression of the coil will be equal to the total gap in the coil.

∴ Maximum compression, $\delta = 2 \times n$ mm

Now using equation (16.27),

$$s = \frac{Cd^4}{64 . R^3 . n} \quad \text{or} \quad 10 = \frac{0.8 \times 10^5 \times d^4}{64 . R^3 . n}$$

$$\therefore \quad d^4 = \frac{10 \times 64 \times R^3 \times n}{0.8 \times 10^5} = \left(\frac{8}{10^3}\right) R^3 \times n \quad \text{...(i)}$$

But from equation (16.28),

Solid length $= n \times d$ or $400 = n \times d$.

$$\therefore \quad n = \frac{400}{d} \quad \text{...(ii)}$$

Substituting the value of n in equation (*i*),

$$d^4 = \left(\frac{8}{10^3}\right) \times R^3 \times \frac{400}{d} = 3.2 \times \frac{R^3}{d}$$

or $$d^5 = 3.2 \times R^3.$$

But mean coil radius,

$$R = \frac{D}{2}$$

$$\therefore \quad d^5 = 3.2 \times \left(\frac{D}{2}\right)^3 = \frac{3.2 \times D^3}{8} = 0.4\, D^3$$

or $$\frac{d^5}{D^3} = 0.4 \quad \text{or} \quad \frac{d^3}{D^3} . d^2 = 0.4$$

or $$\left(\frac{1}{10}\right)^3 . d^2 = 0.4 \qquad \left(\because \frac{d}{D} = \frac{1}{10}\right)$$

$$\therefore \quad d^2 = 0.4 \times 10^3 = 400$$

$$\therefore \quad d = \sqrt{400} = 20 \text{ mm} = \mathbf{2\ cm.} \quad \textbf{Ans.}$$

But $$\frac{d}{D} = \frac{1}{10}$$

$$\therefore \quad D = 10 \times d = 10 \times 2 = \mathbf{20.0\ cm.} \quad \textbf{Ans.}$$

Let us find first number of turns.

From equation (*ii*), we have

$$n = \frac{400}{d} = \frac{400}{20} = 20 \qquad (\because d = 20)$$

$$\therefore \quad \delta = 2 \times n = 2 \times 20 = 40 \text{ mm}$$

We know, stiffness of spring is given by

$$s = \frac{W}{\delta} \quad \text{or} \quad 10 = \frac{W}{40}$$

$$\therefore \quad W = 10 \times 40 = \mathbf{400\ N.} \quad \textbf{Ans.}$$

Using equation (16.24), we have

$$\tau = \frac{16 . W . R}{\pi d^3}$$

$$= \frac{16 \times 400 \times 100}{\pi \times 20^3} \qquad \left(\because R = \frac{D}{2} = \frac{20}{2} = 10 \text{ cm} = 100 \text{ mm}\right)$$

$$= \mathbf{25.465\ N/mm^2.} \quad \textbf{Ans.}$$

Problem 16.42. *Two close-coiled concentric helical springs of the same length, are wound out of the same wire, circular in cross-section and supports a compressive load 'P'. The inner spring consists of 20 turns of mean diameter 16 cm and the outer spring has 18 turns of mean diameter 20 cm. Calculate the maximum stress produced in each spring if the diameter of wire = 1 cm and P = 1000 N.* (AMIE, Summer 1989)

Sol. Given :

Total load supported, $P = 1000$ N

Both the springs are of the same length of the same material and having same dia. of wire. Hence values of L, C and 'd' will be same.

For inner spring

No. of turns, $n_i = 20$

Mean dia., $D_i = 16$ cm $= 160$ mm ∴ $R_i = \frac{160}{2} = 80$ mm

Dia. of wire, $d_i = 1$ cm $= 10$ mm